A Water Tank Drained by Gravity and Filled by Rainfall: A Simple Example of Component-based Modeling

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Abstract

This example was constructed as a simple example based on the *control volume* concept in order to illustrate the basic principles of plug-and-play, or component-based modeling. (The control volume concept lies at the core of fluid dynamics and all conservation laws.) These principles are used by most modern, model coupling frameworks such as CSDMS (Community Surface Dynamics Modeling System) and OMS (Object Modeling System). This example has a simple set of *state variables*, $h(t)$, $V(t)$, $v(t)$ and $Q_{out}(t)$ that describe the state of the system as a function of time. It also has a simple set of *configuration parameters* that are fixed at the start of a model run (in the model's configuration file) and serve to set up the problem. These are A_{top}, A_{out} and h_0. The rainfall rate, R, can be viewed as a driver or forcing variable for the problem, which could be obtained from another model component that is coupled to the water tank model component, or read from a data file. This example is simpler than typical computational models in that there is an analytic solution, so numerical methods of solving differential equations, which can become unstable, are not needed. Also, it doesn't require a discretization of space, so there is no spatial grid, which would introduce additional complexity.

1 Theory

Consider a cylindrical water tank, so that all horizontal cross-sections have the same shape, with an area given by, A_{top}. Assume that the vertical height of the tank, although not specified, is always large enough to accommodate the water depth in the tank, $h(t)$. (That is, assume that "overtopping" is not possible.) Assume also that the top of the water tank is open to the sky, and that there is a small outlet at the bottom of the tank that allows it to drain due to gravity. The volume flow rates (or discharges) into and out of the water tank are given by:

$$Q_{in} = R\,A_{top} \tag{1}$$
$$Q_{out} = v\,A_{out}, \tag{2}$$

where R is the rainfall rate, A_{top} is the top area of the water tank, A_{out} is the cross-sectional area of the tank outlet, and $v(t)$ is the mean flow velocity in the outlet. Since the cross-sectional area of the tank doesn't change in the vertical dimension, the volume of water in the tank at time t is given by $V(t) = A_{top}\,h(t)$. The time derivative of $V(t)$ is therefore given by:

$$\frac{dV}{dt} = A_{top}\,\frac{dh}{dt} = Q_{in} - Q_{out}. \tag{3}$$

When $R = 0$, one can use unsteady conservation of energy (for an inviscid, incompressible fluid) to derive a differential equation that governs the depth of water in the tank, $h(t)$ (see Libii (2003) for more information), namely

$$2\,g\,h = \left(\frac{dh}{dt}\right)^2 \left[\left(\frac{A_{top}}{A_{out}}\right)^2 - 1\right]. \tag{4}$$

This equation results from assuming that the tank is draining slowly, so that the second derivative of $h(t)$ is much smaller than the gravitational constant, g. The closed-form solution to this ODE is

$$h(t) = h_0 \left(1 - \frac{t}{t_d}\right)^2, \tag{5}$$

where t_d is the time to completely drain the tank. Notice that $h(0) = h_0$ and $h(t_d) = 0$. Inserting (5) into (4) and simplifying, we find that

$$t_d = \left(\frac{2\,h_0}{g}\right)^{1/2} \left[\left(\frac{A_{top}}{A_{out}}\right)^2 - 1\right]^{1/2}. \tag{6}$$

For example, if the tank has a top radius of 30 meters, an outlet radius of 5 centimeters and an initial water depth of $h_0 = 1$ meter, it will take 45.15 hours for the tank to drain completely when $R = 0$. Note, however, that if we double the initial depth to 2 meters (keeping other parameters the same), the tank will take 63.85 hours to drain — not twice as long. If the tank did not have an outlet, but $R > 0$, then the time to fill the tank from a depth of 0 to a depth of d_f would be given by $t_f = d_f/R$. Using (3) (with $R = 0$) and computing $h'(t)$ from (5), it can be shown that the $v(t)$ appearing in (2) is given by

$$v(t) = [2\,g\,h(t)]^{1/2}\,\frac{A_{top}}{\left(A_{top}^2 - A_{out}^2\right)^{1/2}}. \tag{7}$$

If $A_{top} \gg A_{out}$, this is closely approximated by

$$v(t) \approx [2\, g\, h(t)]^{1/2}\,. \tag{8}$$

Since the tank is draining slowly, equation (8) can also be derived from *Bernoulli's principle* for steady, potential flow:

$$z_1 + \frac{P_1}{\rho g} + \frac{v_1^2}{2\, g} = z_2 + \frac{P_2}{\rho g} + \frac{v_2^2}{2\, g}. \tag{9}$$

At the top of the water in the tank we have $z_1 = h$ and $v_1 = R \ll v_2$, and at the tank outlet (bottom) we have $z_2 = 0$ and $v_2 = v(t)$. The top and outlet of the tank are both at atmospheric pressure, so $P_1 = P_2$. Inserting these values into (9) results in (8). Combining equations (1), (2), (3) and (8), we therefore find that

$$\frac{dh}{dt} = R - (2\, g\, h)^{1/2} \left(\frac{A_{out}}{A_{top}} \right) = R - F(h), \tag{10}$$

where $F(h) = (2\, g\, h)^{1/2}\, (A_{out}/A_{top})$. This equation shows that if $R > F(h)$, then $dh/dt > 0$, so $h(t)$ will *increase* until $F(h) = R$ and $dh/dt = 0$. This steady-state situation is characterized by a water depth of

$$h_f = \frac{1}{2\, g} \left[R \left(\frac{A_{top}}{A_{out}} \right) \right]^2. \tag{11}$$

For example, if $R = 60$ [mmph] (sustained), $g = 9.81$ [m s^{-2}], $r_{top} = 30$ [m], and $r_{out} = 0.05$ [m], we find that h increases asymptotically from $h_0 = 1$ [m] to a steady-state value of $h_f = 1.83486$ [m].

Similarly, if $R < F(h)$, then $dh/dt < 0$, and $h(t)$ will *decrease* until $F(h) = R$ and $dh/dt = 0$. Here again, h_f is given by (11). For example, if $R = 40$ [mmph] (sustained), $g = 9.81$ [m s^{-2}], $r_{top} = 30$ [m], and $r_{out} = 0.05$ [m], we find that h decreases asymptotically from $h_0 = 1$ [m] to $h_f = 0.81549$ [m].

References

Libii, J.N. (2003) Mechanics of the slow draining of a large tank under gravity, *Am. J. Phys.*, **71**(11), 1204–1207.