Unsaturated Zone Hydrology for Scientists and Engineers

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SATURATED WATER FLOW IN SOIL

INTRODUCTION

This chapter will serve as a brief review of fluid flow in saturated media, for some; for others, it is new material to learn before studying the more complex factors involved in unsaturated fluid flow. There are essentially three types of fluid flow: (1) saturated flow; (2) unsaturated flow; and (3) vapor flow. This chapter focuses on saturated flow; for types 2 and 3, the reader is referred to chapter 8 (unsaturated water flow) and chapter 9 (gaseous diffusion). The driving forces that cause saturated flow are gravitational and pressure-potential gradients. For steady flow in a saturated medium, the change in volumetric water content with respect to time is zero, that is, the water content is equal to the porosity of the medium.

7.1 DARCY'S LAW

The flow of water through water-saturated sand was studied by the French scientist Henry Darcy (1803–1858) in Dijon, France (see Philip 1995). In 1856, in one of the most famous hydrology experiments performed so far, Darcy showed that the volume of water that passes through a bed of sand per unit time is dependent on four things: (1) cross-sectional area of the bed; (2) the bed thickness; (3) the depth of ponded water atop the bed; and (4) on K, the hydraulic conductivity. This is now known as Darcy's law, expressed mathematically as

$$Q = K\frac{(A\,\Delta H)}{L} \tag{7.1}$$

where Q is the volume of water that passes through the bed (or column), cm^3 per unit time; K is the hydraulic conductivity (also called the proportionality constant), in cm per second; A is the cross-sectional area of the column (cm^2); and ΔH is the difference between the head at the inlet boundary and the head at the outlet boundary; these two boundary heads will be discussed later in this chapter. (It should be noted that H is equal to $h + L$, L representing the length of the column or bed thickness, cm.) Since $q = Q/A = V/At$, then equation 7.1 is also expressed as

$$q = \frac{V}{At} = K\frac{(\Delta H)}{L} \tag{7.2}$$

or

$$K = \frac{VL}{At\Delta H} \tag{7.3}$$

where V is the volume of water, cm^3, and t is time, in seconds.

The flux density, q, is the rate of water movement through a medium. To be precise, flux density is the volume of water that passes through a plane perpendicular to the direction of flow, per unit time. Although often referred to as flux, this is incorrect; flux is the volume of water flowing through the medium per unit time, expressed as $Q = V/t$. Flux density (as expressed by Darcy's Law) is dependent on the hydraulic gradient as well as on the type of medium involved. The term filter velocity is sometimes mentioned in conjunction with flux density when the unit cross-sectional area of a specific soil volume is referred to, usually as the mean speed of the soil water through the pore space, described as $q' = q/\theta$, where θ is volumetric water content.

7.2 HYDRAULIC CONDUCTIVITY AND PERMEABILITY

In the previous equations we showed one proportionality constant, K. This is often termed "big" K to represent hydraulic conductivity, which depends on properties of both the fluid and the medium; there is also a "small" k. A difference exists between the two, that we attempt to clarify here, to eliminate any confusion about Darcy's law and the proportionality coefficients associated with it (i.e., big K and small k). If we express Darcy's law as a flux density equation, then

$$q = -K\frac{\partial P_h}{\partial s} = -K\left(\frac{\partial P_t}{\partial s} + \rho_l g\frac{\partial z}{\partial s}\right) \tag{7.4}$$

or

$$q = \frac{k}{\eta}\left(\frac{dP_h}{ds}\right) \tag{7.5}$$

where K is the hydraulic conductivity (m s^{-1}) and k/η is the mobility. Mobility has the units of m^2 Pa^{-1} s^{-1}, and provides the Darcy flux (in cm/s) when multiplied by the gradient in the pressure potential; the hydraulic conductivity (K) provides the Darcy flux when multiplied by the hydraulic head gradient. Mobility can be determined from K by dividing by ρg, noting that s is distance. Thus, units for small k are m^2.

Hydraulic conductivity ($K = k\rho g/\eta$) can be separated into two factors: fluidity (defined as $\eta/\rho g$), and intrinsic permeability. The intrinsic permeability (k) of a medium is a function of pore structure and geometry. We can gain further understanding of the concept of permeability if we optimize the pore space by using capillary-tube models, starting with a single, smooth, straight capillary. For this assumption, Poiseuille's law shows that the rate of discharge for the tube is expressed as

$$Q = -\frac{\pi r^4}{8\eta}\frac{\partial P_h}{\partial s} \tag{7.6}$$

where r is the radius and η is the viscosity of the fluid; all other parameters are as previously discussed. Since $q = Q/A$, and $A = \pi r^2$, then

$$q = -\frac{r^2}{8\eta}\frac{\partial P_h}{\partial s} \tag{7.7}$$

and, by comparison with equation 7.6, $k = r^2/8$.

If the column contains a bundle of capillary tubes or parallel pores of only one size with a certain number of pores, n (perpendicular to flow), then $q = nQ/A$. However, not all pores within a medium are of the same diameter. As a result, n_i represents the number of pores within a system in the ith class with a radius r_i, and using this approach, the flux density is expressed as

$$q = \sum \frac{n_i Q_i}{\pi r^2} = -\frac{1}{8\eta}\frac{\partial P_h}{\partial s}\sum_{i=1}^{n} n_i r_i^2 \tag{7.8}$$

Equation 7.8 is invalid since all pores are not straight, smooth, and parallel to each other. Pore water does not flow in a straight line, but instead travels around individual particles and through varying pore sizes within the medium, resulting in a much longer path than a straight line. The effect of this meandering flow path on the permeability of the medium can be accounted for by tortuosity τ. This is defined as the square of the ratio of the fluid flow path to L, the length of path over which the pressure gradient is effective.

Bear (1972) states that the ratio of $(L/L_e)^2$ (where $L_e > L$) is the correct form to use for tortuosity, rather than L/L_e, as presented in some texts. This is because the effective flow path through a porous medium affects velocity and driving force (which is the hydraulic gradient in cases that involve solution, and pressure in cases that involve air). If we assume a straight path of length L for fluid flow through a soil column (versus the effective length, L_e, which is a meandering path), we can project the flow direction, x, of L_e onto L to obtain an average velocity, $\overline{v}$, in a direction tangential to the axis of the soil capillary or tube of interest. We define this velocity as v_s. Because of the meandering path, even if $|v|$ is constant, v_s can vary. Based on analysis presented by Carman (1937), $\overline{v}$ is defined as the magnitude of the average tangential velocity, and the mean value of v_s is defined as $\overline{v}_s = v(L/L_e)$, which is the velocity component. The absolute value of the mean hydraulic gradient, $|\overline{v}\phi|_s$ (where $\phi = dh/dl$), acts as the driving force in the porous medium; thus, $|\overline{v}\phi|_s = (\Delta\phi/L_e)/(L_e/L)$ and, extending Poiseuille's law to flow in a noncircular tube, $v = (R^2\rho g/m\eta)/(\Delta\phi/L_e)$, where R is the hydraulic radius of the tube and m is a shape factor accounting for the noncircular shape of the tube. Thus,

$$\overline{v}_s = (R^2\rho g(L/L_e)^2|\overline{\nabla}\phi|_x/m\eta = [(R^2\rho g/m)(L/L_e)^2|\overline{\nabla}\phi|_x]/\eta,$$

where $(L/L_e)^2 < 1$ is called the tortuosity factor. The erroneous presentation of L/L_e as the tortuosity factor by some authors arises from failure to account for the effects of L_e on both velocity and the driving force for flow. Values of L/L_e mentioned in the literature vary from 0.56 to 0.8 (Bear 1972).

Because $n_i r_i^2$ is the contribution of $(\Delta\theta)_i$ of the pores with radius, r_i, to the total volume fraction of water (θ), the flux density is also

$$q = -\frac{1}{8\eta\tau}\frac{\partial P_h}{\partial s}\sum_{i=1}^{n}(\Delta\theta)_i r_i^2 \tag{7.9}$$

where $(\Delta\theta)_i = n_i r_i^2$ is the contribution of pores of radius class i to the total volume of water within this fraction. We can compare equation 7.9 to equation 7.4 to obtain the hydraulic conductivity, such that

$$K = \frac{\rho g}{8\eta\tau}\sum(\Delta\theta)_i r_i^2 \tag{7.10}$$

This equation permits an estimation of either hydraulic conductivity or mobility, given the size distribution of pores filled with water.

The most concise way to express tortuosity is $\tau = (L/L_e)^2$, where L_e is effective length (or actual path length) through which a molecule of water must travel through the length (L) of a column. Tortuosity can also be evaluated by measuring the electrical resistivity of a medium, E_s, which assumes the pore space is filled with fluid of known resistivity E and saturated conditions. Then

$$\left(\frac{L}{L_e}\right)^2 = \frac{E}{\phi E_s} \tag{7.11}$$

where ϕ is the porosity (i.e., the area of conducting pore per unit cross-sectional area).

QUESTION 7.1

What is the intrinsic permeability of a medium in which $K = 4.15 \times 10^{-7}$ m^2 Pa^{-1} s^{-1} ? Assume $\eta = 10^{-3}$ Pa s.

QUESTION 7.2

Poiseuille's law as listed in equation 7.7 is a special case of Darcy's law. In Poiseuille's law, what terms correspond to K in Darcy's law?

QUESTION 7.3

Express the hydraulic potential on a per-volume, per-mass, and per-weight basis.

QUESTION 7.4

Express the units for K and k.

QUESTION 7.5

How would you define the volume basis for flux density, for water flow in soil?

7.3 HYDRAULIC CONDUCTIVITY VALUES OF REPRESENTATIVE SOILS

The previous sections have outlined Darcy's law and given the basic equations for water flux. The constant of proportionality of Darcy's law (K) has been termed the hydraulic conductivity, and is a function of both the properties of the medium and the fluid. The hydraulic

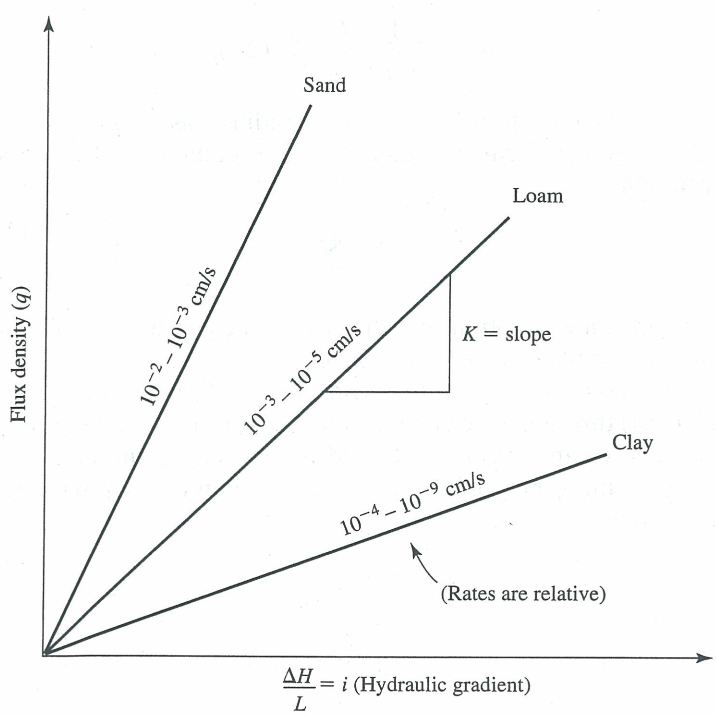

Figure 7.1 Plot of flux density versus the hydraulic gradient for various soil types

conductivity of soils has a wide range; generally, K can range from 10^{-9} cm/s for clay to 1 cm/s for clean sand. Lower values of K for a clay medium (with smaller pore sizes) are likely due to the drag exerted on the viscous fluid by the walls of the pores. A medium with a wide range of pore sizes conducts fluid much more rapidly than a medium with a narrow range of pore sizes; this is especially true if pores, preferential flowpaths, or macropores form continuous paths through the body of the soil.

Particles of smaller-sized individual grains (such as clays compared to sands) have a larger specific surface area, increasing the drag on water molecules that flow through the medium. Thus the result is a reduced intrinsic permeability and K (see figure 7.1).

7.4 FACTORS AFFECTING PERMEABILITY AND HYDRAULIC CONDUCTIVITY

Intrinsic permeability is a property of the medium alone, as the units (L^2) suggest. For soils that do not interact with fluids and change fluid properties or vice versa, this indicates that the same value for k will be obtained for different fluids. However, if fluid–media interactions alter the medium structure, the intrinsic permeability can be altered greatly. For example, a soil with a different composition than that of the native (or residual) water may react with clay minerals in the medium. Such interactions can cause swelling of the clay lattice, thereby reducing the pore space available for flow, or can result in clay disaggregation, allowing clay platelets to migrate and block pore throats.

McNeal and Coleman (1966) evaluated seven soils of varying clay mineralogy and found that decreases in permeability were more pronounced for soils high in 2:1-layer silicates, and that the most labile permeability was exhibited by soils that contained the most montmorillonite. They also found that soils containing considerable amorphous material were more stable than the average soil, and that a soil high in kaolinite and sequioxides was basically insensitive to variation in solution composition.

Petroleum engineers are also concerned with the effects of water composition on permeability, since they often inject water foreign to the formation in order to enhance oil recovery. In laboratory studies, Johnston and Beeson (1945) determined that the permeability of clay-rich (i.e., water-sensitive) formation samples to distilled water can be thousands of times less than to salt water. Originally, petroleum engineers had many problems with waterfloods of petroleum reservoirs, until they learned to use water compatible with the geologic formation.

Permeability can also be affected by the kinetic energy associated with rainfall and irrigation events since this energy disperses particles on the particle surface, causing mechanical crusting that results in lower values of K at the surface. In addition to mechanical crusting, water quality can be a factor in crusting during precipitation, as the nearly distilled rain water contacts the surface sediments and causes the clays to disperse due to double-layer effects (see chapter 3).

A third cause of permeability reduction occurs due to entrapped air. As a previously unsaturated medium is saturated with water, some air is trapped within the system. This trapped air can form bubbles of varying size within the soil, depending on the amount of air present and pore size; such bubbles can obstruct the flow of water.

An additional cause of hydraulic conductivity reduction involves microbial processes. The bacteria themselves may clog the medium, or they may induce chemical reactions that produce clogging slimes. For soils subjected to continuous ponding or submergence, K will decrease initially, due to air entrapment and the leaching of electrolytes, causing dispersion of clay aggregates that seal some pores. Typically, after the initial decrease, an increase in K is seen due to the gradual dissolution of entrapped air caused by water movement. Finally, a reduction of K follows because microbial sealing (the growth of bacterial colonies that clog soil pores) exceeds the rate of increase of K caused by the removal of entrapped air; this results in a gradual, yet continual decrease in K. The reduction of K due to microbial sealing can be significant; it is a major reason why artificial recharge basins are rotated for use.

7.5 LIMITS OF DARCY'S LAW

One of the assumptions of Stokes' law (as discussed in chapter 2) was that the velocity of the particle falling through the solution must be small enough to ensure that flow is always laminar. Similarly, for small-flow velocities such as flux densities or specific rate of discharge (as q is often referred to), the drag forces between the solid surfaces of the medium and the water molecules is proportional to q. Considering the specific surface area of a medium, the area of contact between solid particles and water can be quite large. This means that the force of drag divided by the mass of water increases rapidly with an increase in flow velocity; however, the driving forces acting on the water are not very large. Therefore, the drag force acting on the water reaches a magnitude equal to the driving force. When this occurs, the net force is zero, because each of these forces acts in an opposite direction. As a result, acceleration ceases and a steady flow is achieved until the balance of forces is disturbed; in this situation, inertial forces are insignificant.

As shown by the departure of the "observed" curve from the predicted curve in figure 7.2, Darcy's law does not apply at large enough flow velocities, that inertial forces become significant relative to viscous forces. Darcy's law generally applies when the Reynolds number, defined as the ratio of the inertial forces (as given by the product of fluid density, flux density, and median pore size), to the viscous forces (as given by the fluid viscosity) is less than about 1 (Bear 1972). Darcian flow rates are usually not exceeded in granular media or nonindurated rocks, but flow rates that do exceed the upper limit of Darcy's law (as shown in figure 7.2) are common in karstic limestones and dolomites, as well as in cavernous volcanics.

For soils with low permeability such as clays (see figure 7.2, lower curve), deviations from Darcy's law have been observed, especially at low gradients. Low hydraulic gradients

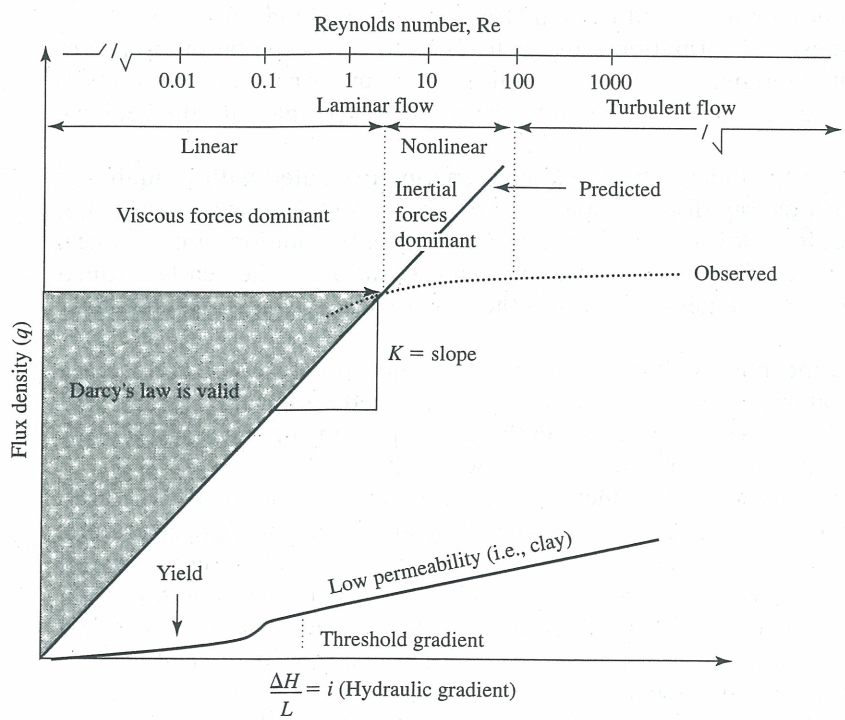

Figure 7.2 A schematic classification of flow through soil, illustrating the Reynolds number, flow conditions under which Darcy's law is valid, and predicted versus observed plots for K.

can cause no-flow conditions, or flow rates that are so low that they are less than proportional to the hydraulic gradient. This likely results from water molecules that are subjected to adsorptive forces near the solid particles, which means the water near the particle is more "rigid" than bulk water within the medium. Such water exhibits properties associated with a "Bingham liquid" (which has a yield value), rather than a "Newtonian liquid." A good example of bound (or rigid) water is hygroscopic water, which develops a structure similar to that of ice. Certain media have a threshold gradient (figure 7.2, bottom curve), below which the flux density is lower than that predicted by Darcy's law, and usually very near zero. In these cases, the flux density is proportional to the hydraulic gradient only when the gradient value exceeds the threshold value. Olsen (1966) showed that (except for montmorillonite clays), deviations from Darcy's law at the low-range are generally experimental artifacts; also, the likely cause of non-Darcy flow is the quasi-crystalline structure of water near the clay surface, and reversible clay fabric changes induced by seepage forces. The two characteristics governing the extent to which these mechanisms cause deviation from Darcy's law are pore size and rigidity of the clay fabric.

Considering the previous discussion, a more general form of Darcy's law can be written, such that

$$q = -K\left(\frac{\Delta H}{\Delta L}\right)^m \tag{7.12}$$

where m would equal 1 for most flow situations in which Darcy's law could be applied (Ghildyal and Tripathi 1987). If m is greater than 1, laminar flow does not apply, and Darcy's law should not be used (see fig. 7.2).

7.6 DARCY'S LAW AND WATER FLOW THROUGH SOIL COLUMNS

A common practice has been to utilize the principles associated with Darcy's law on many small-scale studies, especially those laboratory soil columns that allow scientists to determine hydraulic conductivity in an economic, efficient, and controlled manner. While using manometers for measuring H in soil columns and in-situ has been a common practice for several decades, recent technological advances allow the use of pressure transducers for more accuracy and automation of sequential measurements.

Flow within a soil profile (or system) can occur in any direction, whether vertical or horizontal. When the hydraulic heads H_1 and H_2 are measured by manometers labeled 1 and 2 (see figure 7.3), Darcy's equation is expressed as

$$q = K\frac{(H_1 - H_2)}{L} \quad \text{or} \quad q = K\frac{(h_1 - h_2) + (z_1 - z_2)}{L} \tag{7.13}$$

If the manometers in figure 7.3 are replaced by pressure transducers, the pressure potential, P, for the two transducers will be $P_1 = P_1 + \rho g z_1$, $P_2 = P_2 + \rho g z_2$, and

$$\Delta P = [P_1 - P_2 + \rho g(z_1 - z_2)] \quad \text{or} \quad \Delta H = \left[\frac{(P_1 - P_2)}{\rho g}\right] + (z_1 - z_2) \tag{7.14}$$

QUESTION 7.6

Using the soil column shown in figure 7.3, what is the volume of water, V, that will flow through the soil in 30 minutes? Assume $K = 3.52$ cm/hr, $L = 70$ cm, cross-sectional area $= 7.5$ cm^2, $z_2 = 0$, $h_1 = 10$ cm, $h_2 = 5$ cm, and the angle of incline for the column is 45 degrees.

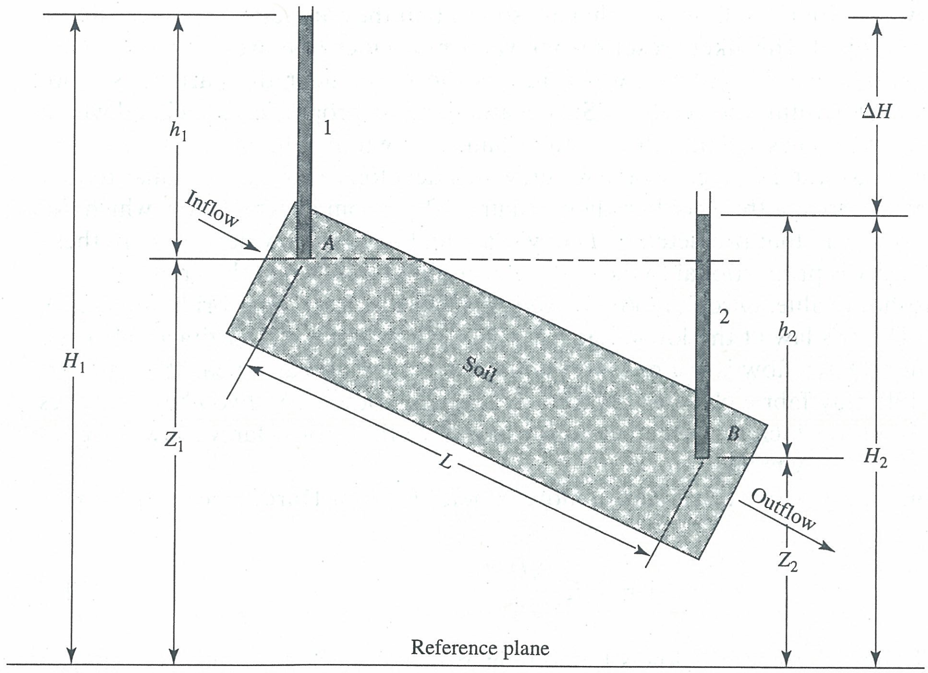

Figure 7.3 Water flow through an inclined column

7.7 HOMOGENEITY, HETEROGENEITY, ISOTROPY, AND ANISOTROPY OF SOILS

Models of water flow that are based on analytical equations as solutions to the Laplace or diffusion equation treat a medium as homogenous—that is, the same spatial properties exist throughout the medium. In this case, K would be independent of position; however, due to spatial variability, K is dependent on physical properties at any given position within a medium, and as such, the medium is considered heterogeneous.

Because K values often vary by more than two orders of magnitude within the same soil profile or hydrogeologic unit, an arithmetic mean appears to give more emphasis to the higher values of K. A more representative description of the average K value of the system is the geometric mean. We determine the geometric mean by taking the natural log of each value, find the mean of the natural logs, and then obtain the exponential (e^x) of that value (Fetter 1994; Freeze and Cherry 1979). Statistical methods are usually performed on media parameters in order to obtain more information about them than either arithmetic or geometric means provide.

In general terms, statistical distributions can provide quantitative descriptions of a medium and therefore, the degree of heterogeneity *for* that medium. A well-known fact is that the probability density function, *pdf*, for hydraulic conductivity (K) often fits a log–normal function, for which log K has a normal distribution. Freeze (1975) showed that the standard deviation of log K (independent of the units of measurement) ranges from approximately 0.5 to 1.5. Additionally, Greenkorn and Kessler (1969) state that a homogenous formation has a *pdf* of K that is monomodal, while for a heterogeneous medium, the *pdf* of K is multimodal; and for the homogenous medium, K varies only slightly in spatial terms, but has a constant mean K throughout the medium.

In most natural-aquifer systems comprised of unconsolidated sediments or sedimentary rocks, the geometry of the void space varies with direction, due to the preferred orientation of plate-shaped clay particles within the medium. Because of these pore-geometry variations, the hydraulic conductivity of the medium varies with direction, and is termed "anisotropic." Permeability is usually greatest when parallel, and least when perpendicular, to the plate orientation. For systems in which flow is three-dimensional, the medium exhibits three permeability axes, including those where permeability is greatest and least (the principal axes), and a third that is orthogonal to the principal axes.

By generalizing the one-dimensional form of Darcy's law for a medium that is anisotropic, we can describe flow in three dimensions. For three-dimensional flow, the velocity (v) is a vector with components in the x, y, and z directions. The velocity for each direction can be expressed by

$$v_x = -K_x \frac{\partial h}{\partial x}$$
$$v_y = -K_y \frac{\partial h}{\partial y} \tag{7.15}$$
$$v_z = -K_z \frac{\partial h}{\partial z}$$

where K_x is the hydraulic conductivity in the x direction. Note that we use partial derivatives here, since h is a function of x, y, and z. For generalized flow in three dimensions for the x direction (Freeze and Cherry 1979), we may write

$$v_x = -K_{xx} \frac{\partial h}{\partial x} - K_{xy} \frac{\partial h}{\partial y} - K_{xz} \frac{\partial h}{\partial z}$$
$$v_y = -K_{yx} \frac{\partial h}{\partial x} - K_{yy} \frac{\partial h}{\partial y} - K_{yz} \frac{\partial h}{\partial z} \tag{7.16}$$
$$v_z = -K_{zx} \frac{\partial h}{\partial x} - K_{zy} \frac{\partial h}{\partial y} - K_{zz} \frac{\partial h}{\partial z}$$

Thus, we obtain nine components of K for the most general case. In matrix form, these three equations form a second-rank tensor, the hydraulic conductivity tensor (Bear 1972). If the principal directions of anisotropy coincide with the x-, y-, and z-coordinate axes, equation 7.15 can be used instead of equation 7.16. In many (but not all) situations, it is possible to choose the coordinate system that will satisfy this condition. A situation in which it may not be possible is a heterogenous anisotropic system, where the direction of anisotropy varies from one location to the next.

7.8 SATURATED FLOW IN LAYERED MEDIA

For heterogeneous media, K varies in spatial terms and is commonly modeled to predict water movement through a layered medium, assuming that each layer is homogenous but the medium itself is heterogeneous. The average K depends on the direction of flow: for horizontal flow, the average K is the arithmetic mean; the vertical K is defined as the harmonic mean, generally smaller. Thus, a heterogeneous, layered medium can be modeled as an anisotropic homogeneous medium in which $K_H = \Sigma K_i b_i / \Sigma b_i$ and K_z = eq. 7.17. Considering the fact that many geologic media such as sedimentary rocks and marine deposits contain depositional layers, this approach is reasonable. Each layer will have a different hydraulic conductivity denoted as K_1, K_2, etc.

Soils are often comprised of several layers of thickness d. Each layer is generally considered homogeneous and will have a varying hydraulic conductivity, as illustrated in

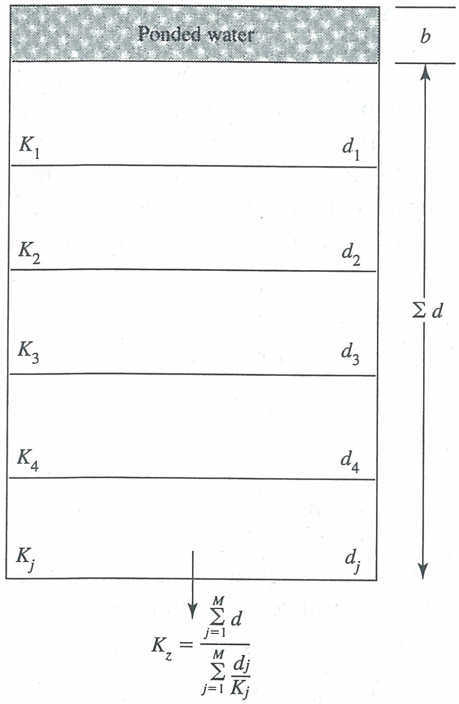

Figure 7.4 Illustration of calculation of K_z through a layered-soil profile (see equation 7.14); K is the hydraulic conductivity for layer 1, . . . , j; d is layer thickness for layers 1, . . . , j; and K_z is the effective hydraulic conductivity through the entire profile

figure 7.4. If the medium were composed of one layer, the vertical hydraulic conductivity could be simply calculated using equation 7.3. However, for a layered medium the effective vertical K will depend on resistance to flow through each layer. As a result, one obtains a hydraulic resistance, R (derived from Ohm's law), such that $R = d_j/K_j$, where d_j is the thickness of each layer and K_j is the hydraulic conductivity of each layer. Consequently, we can write an expression for K as a sum of all layers and obtain the effective vertical hydraulic conductivity such that

$$K_z = \frac{\sum_{j=1}^{m} d}{\sum_{j=1}^{m} \left(\frac{d_j}{K_j}\right)} \tag{7.17}$$

where K_z is the effective hydraulic conductivity; the summation of d in the numerator is the thickness of the entire medium; and the hydraulic resistance is the denominator, summed for each layer assuming perpendicular flow.

Use the following steps to obtain the effective hydraulic conductivity and the flux density, q, for a layered profile: (1) define a reference point (elevation); (2) determine the effective hydraulic conductivity using equation 7.17; (3) using the reference elevation, select two points (1 and 2) where the hydraulic head (H) is known, and determine the gradient, $\Delta H/\Delta d$, then solve for $q = -K[\Delta H/\Delta d]$; and (4) substitute the values obtained for q and apply Darcy's law across each layer to calculate the pressure at the layer interfaces. As previously explained (and assuming K is constant), we should expect the pressure to change linearly within each layer.

QUESTION 7.7

You are working in the Florida Everglades, which has just been ravaged by a major hurricane. A large area around a municipal wastewater treatment plant in a small town has 100 cm of ponded water covering the land surface. State health officials fear that unless the water drains rapidly, a severe health hazard will be created for the nearby town; they want you to tell them how long it will take the water to subside. Additionally, they need you to calculate the flux density, total potential, and pressure potential (weight basis) at the interface between each layer, in the soil report they provided to you. Based on soil reports of the area, there is a surficial aquifer about 1 m below the surface which is comprised of two 50-cm layers of soil with different hydraulic conductivities. Therefore, you must consider two cases of saturated flow down through a two-layer medium. The conductivity of layer 1 is 10^{-4} cm/s, and K for layer 2 is 10^{-5} cm/s. These parameters are true for only 60% of the submerged area; for the remaining 40%, K for each layer is reversed (i.e., the less conductive medium overlies the more conductive medium).

QUESTION 7.8

In question 7.7, what causes the difference in sign of the interface pressure potentials?

QUESTION 7.9

A soil column has a 2-mm thick crust at the top and a total length of 20 cm, thus the remaining length of column is 19.8 cm. Water is kept ponded on the surface to a depth of 1 cm and steady-state flow of water is occurring through the column; the column is open to the atmosphere at the bottom. The saturated K of the crust is 0.001 cm/hr and the conductivity of the underlying soil layer is 5 cm/hr. **(a)** What is the effective hydraulic conductivity of the column? **(b)** What is the flux density? **(c)** What is the pressure pontenial on a weight basis at the interface between the crust and the underlying soil layer?

7.9 LAPLACE'S EQUATION

Steady-state saturated flow in three dimensions is usually described by the Laplace equation. Steady-state flow occurs when the magnitude and direction of the flow velocity, at any point in a flow field, are constant with time. As previously mentioned, for steady-state flow to occur through a medium, the law of conservation of mass must apply. This implies that, when considering a specific volume element of interest, the rate of fluid flow into the volume must equal the rate of fluid flow out of the volume element; simply stated, inflow − outflow = 0. The continuity equation relates this law in mathematical notation such that

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = 0 \tag{7.18}$$

The ρv terms have dimensions of mass rate of flow per unit cross-sectional area. For constant density fluid, the expression can be further simplified to

$$-\frac{\partial(v_x)}{\partial x} - \frac{\partial(v_y)}{\partial y} - \frac{\partial(v_z)}{\partial z} = 0 \tag{7.19}$$

A simplified expression of Darcy's law in differential notation is $v = -K\,dh/dl$; substituting Darcy's law for v in the above equation for all x, y, and z, we obtain

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = 0 \tag{7.20}$$

This equation applies to conditions of steady-state flow through saturated anisotropic media under the condition that the principle axes of flow coincide with those of the K tensor. If we assume a homogeneous, isotropic medium, the equation can be simplified to

$$\frac{\partial^2(h)}{\partial x^2} + \frac{\partial^2(h)}{\partial y^2} + \frac{\partial^2(h)}{\partial z^2} = 0 \qquad (7.21)$$

This last equation is termed Laplace's equation, the solution of which is a function $h(x,y,z)$ that gives the value of the hydraulic head, h, at any point within a three-dimensional volume, subject to specified boundary conditions. Solutions of Laplace's equation allow us to plot flow nets, flowlines, and equipotential maps of h. The Laplace equation usually assumes that all flow is from water stored in the aquifer; however, field research often shows significant flow generated from leakage into the aquifer through overlying or underlying confining layers.

The boundary conditions specified in equation 7.21 enable us to obtain a numerical solution to a flow problem. If steady horizontal flow through a homogeneous, isotropic porous medium is assumed, Laplace's equation can be written as

$$\frac{\partial^2(h)}{\partial x^2} + \frac{\partial^2(h)}{\partial y^2} = 0 \qquad (7.22)$$

Equation 7.22 is referred to as the equation of flow for steady-state saturated in flow in the xy plane. Consider the simple problem of flow through a rectangle bounded by equipotentials at $x = 0$ and $x = x_L$, and by impermeable boundaries at $y = 0$ and $y = y_L$. For this simplified problem, the mathematical statement of the boundary condition, given by Freeze and Cherry (1979), is

$$\begin{aligned} \frac{\partial h}{\partial y} &= 0 \quad \text{on} \quad y = 0; \quad y = y_L \\ h &= h_0 \quad \text{on} \quad x = 0 \\ h &= h_1 \quad \text{on} \quad x = x_L \end{aligned} \qquad (7.23)$$

where $h = h_0$ at $x = 0$ and $h = h_1$ at $x = x_L$ and x is the direction of flow; the y direction represents plane thickness that would extend from $y = 0$ at $x = 0$ to $y = y_L$. When boundary conditions are specified, the problem becomes a boundary-value problem, in actuality a mathematical model. The mathematical model has a four-step process of analysis: (1) examine the physical problem; (2) specify boundary conditions and replace the physical problem with a mathematical statement; (3) solve the mathematical statement; and (4) interpret the results in terms of the original physical problem. For any problem that involves steady flow through a confined or water-table aquifer, specified boundary conditions allow us to obtain a solution to the problem via the four basic steps listed above. A more detailed discussion of boundary conditions and boundary value problems is given in Bear (1972), Freeze and Cherry (1979), Bear and Bachmat (1991), and Smith (1985).

The water table is defined as the phreatic (or free) surface in an unconfined aquifer or confining bed, at which pore-water pressure is atmospheric. The pressure (pressure head) at the water table is zero, thus at any given point along the water table, the hydraulic head is equal to height of elevation of that point from the reference plane; see figure 7.5. The slope of the water table is equal to $\Delta H/S$ or tan θ (see figure 7.5); the hydraulic gradient is given by $\Delta H/L$ or sin θ. Also note in figure 7.5 that the saturated zone extends to the top of the water table, and that above the saturated zone is the capillary fringe, the thickness of which depends on the pore-size distribution of the media—that is, the finer the medium, the greater the capillary fringe's extent.

In a water-table situation, flow is in fact three-dimensional. Suppose we have a water-table aquifer with a horizontal impermeable base that discharges to a full penetrating stream, as seen in figure 7.6. Equipotentials are perpendicular to both the impermeable base

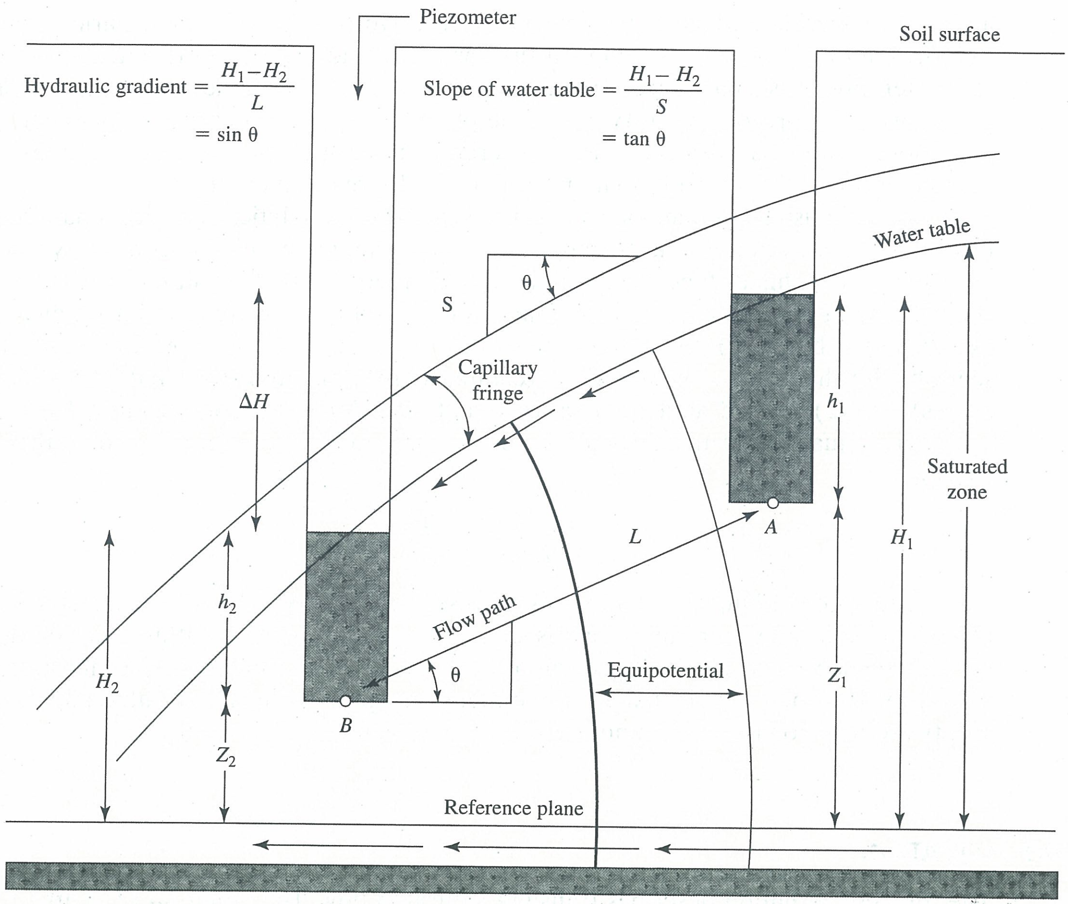

Figure 7.5 Illustration of hydraulic gradient showing differences between gradient and slope of water table

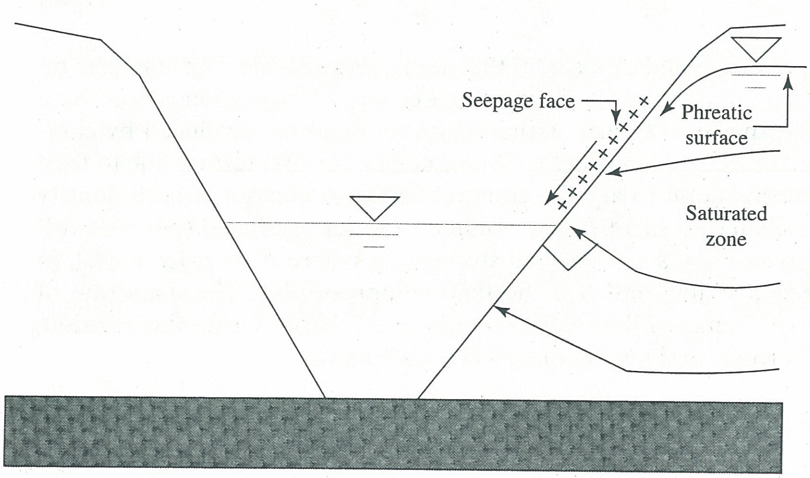

Figure 7.6 Water-table aquifer with horizontal impermeable base discharging to full penetrating stream (after Bear and Bachmat 1991)

and the curving free surface, so they must bend; the hydraulic gradient then varies from being the tangent of the water-table slope at the base of the aquifer to being equal to the sine at the water table. A seepage face is required wherever the water table discharges to a stream, ocean or lake shore, or a well. Typically, the phreatic surface represents a nonlinear boundary whose location is a priori unknown. Therefore, in most cases analytical solutions to free-surface problems are difficult, so numerical methods are often employed.

To avoid using a numerical solution, we can derive analytically approximate solutions based on linearization of the boundary conditions and/or nonlinear continuity equations describing unconfined flow. An example of such methods is the Dupuit approximation, a special case of the Laplace equation that is a powerful tool for treating unconfined flow. It assumes two things: (1) the equipotential extends vertically down from the point of intersection with the water table and the streamlines are horizontal (for small water-table gradients); and (2) the hydraulic gradient is equal to the slope of the water table. The resulting continuity equation for an isotropic, homogeneous water-table aquifer based on the Dupuit assumption is:

$$\frac{\partial(h^2)}{\partial x^2} + \frac{\partial^2(h^2)}{\partial y^2} = 0 \tag{7.24}$$

Although the Dupuit assumptions represent an approximate solution, results are usually accurate if the water-table slope is small. In addition, the Dupuit assumption does not allow a seepage face, and will give erroneous results in the immediate vicinity of such a boundary. The end result is that a three-dimensional problem can be treated approximately as a two-dimensional one, with no complicated seepage face boundary.

7.10 DIFFUSION EQUATION

The diffusion equation describes transient saturated flow. The equation of continuity is no longer equal to zero as in the case with steady-state saturated flow, but takes the form

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial(\rho\phi)}{\partial t} \tag{7.25}$$

which, by expansion of the right-hand side, yields

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \phi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\phi}{\partial t} \tag{7.26}$$

where ρ and ϕ refer to water density and porosity of the media, respectively. The first term on the right-hand side of the equation is the mass rate of water due to expansion caused by a change in the water density; the second term is the mass rate of water produced by compaction of the soil as a result of change in porosity. Consequently, the first term is due to fluid compressibility and the second is due to aquifer compressibility. A change in both density and porosity results from a change in head, h. The volume of water produced (per unit volume and unit-head change) is termed the specific storage, S_s, where $S_s = \rho g(\alpha + \phi\beta)$, in which α is the aquifer compressibility and β is the fluid compressibility. The mass rate of water produced (time rate of change of fluid mass storage) is $\rho S_s\, \partial h/\partial t$. Using this relation, we apply the chain rule of calculus and insert Darcy's law, and obtain

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t} \tag{7.27}$$

which is the equation that describes transient flow through saturated anisotropic media. For transient flow through saturated, homogeneous, and isotropic media, this equation may be reduced to

$$\frac{\partial^2(h)}{\partial x^2} + \frac{\partial^2(h)}{\partial y^2} + \frac{\partial^2(h)}{\partial z^2} = \frac{S_s}{K}\frac{\partial h}{\partial t} \tag{7.28}$$

This is the diffusion equation, first developed by Jacob (1940). The solution $h(x,y,z,t)$ describes the value of h for any point in the flow field at time t. In addition to specifying boundary conditions (as in the Laplace equation), we must also specify initial conditions to obtain a solution for the diffusion equation. The following parameters must also be known: fluid density, ρ; fluid compressibility, β (fluid parameters); hydraulic conductivity, K; aquifer compressibility, α; and porosity, ϕ (aquifer parameters). This equation takes on different forms for confined aquifers, unconfined aquifers, and other situations.

SUMMARY

In this chapter, we discussed Darcy's law and its application to saturated soils and how a hydraulic gradient is defined as a change in total head (H), with a change in distance through the medium in the direction that yields a maximum rate of decrease in head. We also demonstrated that intrinsic permeability is a property of the medium alone; this indicated that, for soils not interacting with fluids and change fluid properties or vice versa, the same value for k is obtained for different fluids. We defined flux density as the amount of water passing through a plane perpendicular to the direction of flow per unit time. We also discussed how Darcy's law does not apply at flow velocities large enough that inertial forces become significant relative to viscous forces (i.e., the limits of Darcy's law).

Additionally discussed was how Darcy's law applied to laboratory work with small soil columns. Definitions of homogeneity, heterogeneity, isotropy, and anisotropy of soil were presented, as well as generalized flow forms of Darcy's law in three-dimensions. We also explained how K varied in spatial terms, commonly modeled to predict water movement through layered soils, using the assumption that each layer was homogeneous. We ended the chapter with discussions of Laplace's equation that described steady-state saturated flow in three dimensions as well as the diffusion equation, that described transient saturated flow. The next chapter presents the fundamentals of water flow in unsaturated soils.

ANSWERS TO QUESTIONS

7.1. Using equation 7.12, $k_i = \eta K = 4.15 \times 10^{-10}\ \text{m}^2$. Of course, this is only a rough estimate, without knowing the tortuosity.

7.2. We simply need to express each law in terms of Q and remember several basics: $A = \pi r^2$ (we use the term P for pressure here, instead of ∂h); $P = \rho g h$; and that ρ and g are constant. For Darcy's law, $Q = K[(A\Delta H)/L]$, and since $A = \pi r^2$ and $P = \rho g h$, we can insert these values into equation 7.7. Hence, for Poiseuille's law, $Q = Ar^2\Delta\rho g h/8\eta L$, which is also expressed as $Q = Ar^2\rho g\Delta H/8\eta L$ (because ρ and g are constant). Rewriting, $Q = [(A\Delta H)/L][(\rho g r^2)/8\eta]$; consequently, the Darcy's law $K = [(\rho g r^2)/8\eta]$ of Poiseuille's law.

7.3. On the basis of volume, the hydraulic potential is expressed as $\text{J m}^{-3}\ \text{m}^{-1} = \text{N m}^{-3}$, which is the force divided by volume. The hydraulic potential per unit mass is $\text{J kg}^{-1}\ \text{m}^{-1} = \text{N kg}^{-1}$, which is force divided by mass. The hydraulic potential per unit weight is J N^{-1} m^{-1} = N N^{-1}, or m m^{-1} (the hydraulic head gradient).

7.4.

$$K = \left(\frac{q}{\frac{\partial h}{\partial x}}\right) = \frac{ms^{-1}}{mm^{-1}} = ms^{-1}$$

$$\frac{k}{\eta} = \left(\frac{q}{\frac{\partial P_h}{\partial x}}\right) = \frac{ms^{-1}}{J\,m^{-2}m^{-1}} = \frac{ms^{-1}}{Pa\,m^{-1}} = m^2\,Pa^{-1}s^{-1}$$

7.5. On a volume basis, the flux density (q) is simply the volume of water flowing through a cross-sectional area of interest perpendicular to the direction of flow per unit time. This is expressed in the units for the last equation, for "little" k in answer 7.2.

7.6. One problem we have is that we do not know z_1; however, we do know that the angle of the column is 45 degrees, thus z_1 is equal to the length of the column multiplied by the sine of the angle. Hence, z_1 = 70 cm × sin 45 = 49.5 cm. Since we are trying to determine V, then we must assume that $q = V/At$. As a result we can set up the problem such that $V/At = K[\{(h_1 + z_1) - (h_2 + z_2)\}/L]$. Inserting both the given and calculated values, we have $V/(7.5\ cm^2)(0.5\ hr)$ = 3.52 cm/hr[{(10 + 49.5) − (5 + 0)}/70]; $V/3.75\ cm^2\ hr$ = 2.74 cm/hr. Thus, V = 10.28 cm^3. There are many ways to calculate q, V, and K. What formula would you obtain if you solved for K? Since the numerator of the calculation above is really ΔH, we obtain equation 7.3. If we use pressure transducers instead of manometers, the solution is simpler.

7.7. To begin with, we need to set up two equations: one for flow through the top layer, and another for flow through the bottom layer; the pressure potential (h) at the interfaces is an unknown in both equations. Discharge through the top layer must equal discharge through the bottom layer. Thus, for Case 1 we have

$$q = K\left[\frac{(h_1 + z_1) - (h_2 + z_2)}{L}\right]$$

$$10^{-4}\ cm\ s^{-1}\left[\frac{(100 + 100) - (h_2 + 50)}{50}\right] = 10^{-5}\ cm\ s^{-1}\left[\frac{(h_1 + 50) - (0 + 0)}{50}\right]$$

$$50 \times 10^{-4}(150 - h_2) = 50 \times 10^{-5}(h_1 + 50)$$

$$\frac{0.725}{0.0055} = h = 131.82\ cm$$

$$q = 10^{-4}\left[\frac{(200 - 132.82 - 50)}{50}\right] = 3.64 \times 10^{-5}\ cm\ s^{-1}$$

Notice that we list h as both h_1 and h_2; this is to keep them initially separate, to avoid confusion. The total potential in this case is simply h + 50 cm = 181.82 cm. To solve for Case 2, put 10^{-5} on the left side of the equations above and 10^{-4} on the right side (i.e., reverse their order), and perform the calculations in the same order to find h = −31.82 cm with a total potential of 18.2 cm. For Case 2, $q = 10^{-5}$ [(150) − (−31.82))/50] = 3.64 × 10^{-5} cm/s, which is the same as for Case 1. Now, using the flux density for determination of water subsidence, we obtain 3.14 cm/d, neglecting loss by evaporation. Thus, the 1-m depth of water will take 31.8 days to subside, causing significant health hazards.

7.8. The difference between the sign in the pressure heads in question 7.7 is simply caused by a difference in K between the heads.

7.9. **(a)** We can use equation 7.17 to obtain the effective hydraulic conductivity:

$$K_z = \frac{\sum_{j=1}^{m} d}{\sum_{j=1}^{m}\left(\frac{d_j}{K_j}\right)} = \frac{20}{\frac{0.2}{0.001} + \frac{19.8}{5}} = 0.098 \text{ cm/hr}$$

(b) The flux density q may be obtained by:

$$q = -K\frac{\Delta H}{L} = -0.098\left(\frac{(h_i + z_i) - (h_o + z_o)}{L}\right)$$

$$= -0.098\left(\frac{(1 + 20) - (0 + 0)}{20}\right) = -0.103 \text{ cm/hr}$$

where subscripts i and o represent parameters at the inlet and outlet boundaries.

(c) The pressure potential we denote as P_3 may be calculated in the same manner as in part **(b)**, but using the saturated hydraulic conductivity of the underlying layer such that

$$q = K_s\left(\frac{(H_3 - 0)}{L}\right) = -0.103 = -5\left(\frac{(P_3 + 19.8)}{19.8}\right)$$

we must now solve for P_3, which yields $P_3 = -19.39$ cm.

ADDITIONAL QUESTIONS

7.10. The saturated hydraulic conductivity of a soil, K_{sat}, has been measured at 4.3×10^{-8} m^2/Pa · s. What is the intrinsic permeability, K_i, for this soil? Assume $\eta = 10^{-3}$ Pa · s.

7.11. What is K_i for a poorly conductive soil?

7.12. You have a soil column in the lab with a soil depth (z), equal to 0.80 m, and a ponded head (H) equal to 0.10 m. What is K_{sat} for this column? Assume a discharge rate of 1.5 cm^3/min and a cross-sectional area of 20 cm^2.

7.13. You are investigating saturated flow within a soil profile with layers of varying hydraulic conductivity; the flux density is the same in all layers. Why?

7.14. For question 7.3, $K_{sat} = -1.185 \times 10^{-5}$ m/s. Assume $z = 0$, $H = 0$, $z = 0.2$ m at $H = 0.6$ m, and $\Delta H = 0.6$ m. What is the flux density?

7.15. You are working with a sugarcane producer who has just harvested his crops and says he must maintain saturation of this soil for 14 days to control root-knot nematodes. Using the answer in question 7.6 (5.92×10^{-6} m/s or 5.12×10^{-1} m/day), how much water will be required over a 14-day period to satisfy this requirement?

7.16. You have a soil profile consisting of two layers. Layer 2 is 0.2 m thick and the bottom layer overlies evenly spaced drains. Since the drainage system provides atmospheric pressure, what is the flux density for each layer? Assume $h = 0$ at $z = 0.2$ m for layer 2; for layer 1, $h = 0.7$ m at $z = 0.6$ m; $K_1 = 1.18 \times 10^{-5}$ m/s and $K_2 = 1/6\ K_1$.

7.17. Suppose you have a two-layered soil with hydraulic characteristics similar to those listed in question 7.6, with a flux density of 5.92×10^{-6} m/s, and this value reduces to -1.97×10^{-6} m/s (q_2 in question 7.7). Using the drainage set-up from question 7.7, will the drainage system be effective in reducing percolation loss through layer 2?

7.18. You have set up a field project that has a two-layered soil profile. The upper layer is 0.10 m and the lower layer is 0.9 m. The lower layer overlies a bed of coarse gravel and K_{sat} of the lower layer is 4.0 times greater than that of the upper layer. Assuming $K_1 = 1.1 \times 10^{-5}$ m/s and $dH_2/dz_2 = 0.875$, what is the flux density to the gravel layer?

7.19. In the text, we showed that at a constant static force (F^s) we have

$$\frac{F^s}{\mathrm{m}} = -\lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s} = -\frac{d\psi}{ds}$$

We also discussed that the corresponding potential in the case of water flow in soil is ψ_h and thus,

$$\frac{\sum F^s}{\mathrm{m}} = -\frac{\partial \psi_h}{\partial s}.$$

In addition, we showed that for the water flow in soil, the driving forces as divided by volume and by weight (respectively) were $-\partial p_h/\partial s$ and $-\partial H/\partial s$. For all normal water flow in soil, the average terminal velocity is much smaller than the value at which the drag force is no longer proportional to the velocity. Thus, when $\Sigma F^d = -\Sigma F^s$, the terminal velocity is constant, and proportional to $-\Sigma F^d/\mathrm{m} = \Sigma F^s/\mathrm{m}$. Using this information and the fact that the flux density (q) is proportional to terminal velocity: **(a)** define the flux density on a volume basis for the water flow in soil and state its dimension (unit); and **(b)** obtain (i.e., derive) the flux density equations (also called Darcy's law) on a weight and volume basis.